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AN INFINITE PLATE WEAKENED BY A CIRCULAR HOLE

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ABSTRACT: *This paper presents a series solution of the problem of an infinite plate weakened by a circular hole. The series method is a common method in the theory of elasticity, since it is very convenient to apply and to solve problems of circular boundaries such as a circular disc, or an annulus under uniform internal and external pressures. A unidirectional tension in the plate is imposed in the direction of the x-axis and the stress distribution around the edge of a circular hole is studied.*

Keywords: *Infinite plate, Series method, Unidirectional tension, Dislocation.*

I. INTRODUCTION

There exist various methods to solve the boundary value problems of the plane theory of elasticity such as power series method, method based on properties of cauchy integrals or direct method of solution and method based on continuation. The series method of solution is directly applicable to the regions bounded by one or two concentric circles. The series method of solution forms a straight forward and efficient technique for the solution of simple stress and displacement boundary value problems. Many problems have been solved using this method by Love (1934). This method has also been used by Timoshenko and Goodier (1951). The problem of an infinite plate when a concentrated force has been applied at a point of the plate has been solved by Muskhelishvili (1963). Some of the problems have been solved by Sokolnekoff (1956) using series method of solution. Although the series method of solution is less effective for more complicated boundary conditions, it can also be used for simple mixed boundary conditions. When the series method of solution is applied to mixed boundary conditions, the problem reduces to the solution of a system of dual series equations. Some problems of mixed boundary conditions by series method of solution have been examined by Cooke and Tranter (1959). Some other problems of mixed boundary conditions by series method of solution have been discussed by Sneddon (1966). Gdoutos et al. (1987) have studied the two-dimensional problem of an infinite plate weakened by a curvilinear hole. Another problem of infinite plate weakened by a hole having arbitrary shape has been discussed by Abdou and Khar-El din (1999). Dowaikh (2004) has solved fundamental problems for an elastic plate weakened by a curvilinear hole. Abdou and Monaquel (2011) have presented an integral method to determine the stress components of stretched infinite plate with a curvilinear hole.

In this chapter the first fundamental problem for an infinite plate has been discussed. The plate is weakened by a circular hole. There is an unidirectional tension in the plate parallel to X-axis. It is assumed that there is no dislocation, because for an infinite plate with one hole rotational dislocation is impossible. Stress distribution has been studied around the edges of a circular hole.

II. FORMULATION OF THE PROBLEM

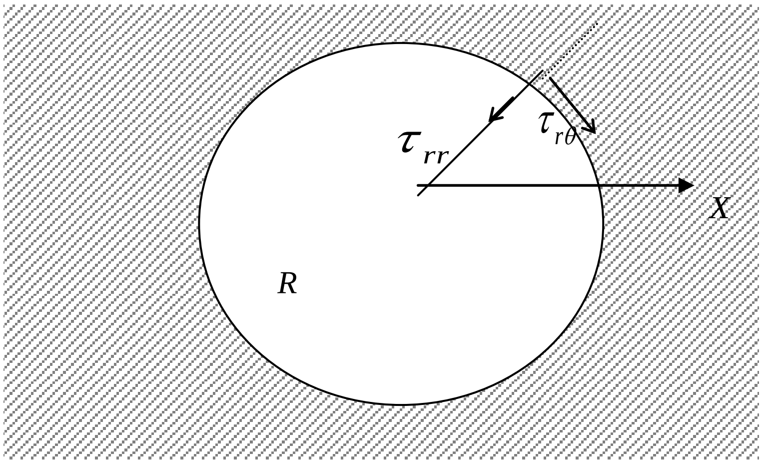
The state of stress and displacement can be represented by means of the two complex functions $\phi(z)$ and $\psi(z)$. Let us draw a circle about the origin as centre with radius R, sufficiently large (Fig. 1). For every point outside L, $\Omega(z)$ and $\omega(z)$ can be expressed using Muskhelishvili (1963).

$$\Omega(z) = -\frac{X+iY}{2\pi(1+\chi)}\log z + \Omega^*(z), \tag{1}$$

$$\omega(z) = \frac{\chi(X-iY)}{2\pi(1+\chi)}\log z + \omega^*(z), \tag{2}$$

where $\Omega^*(z)$ and $\omega^*(z)$ are functions holomorphic outside L except at infinity. A function will be called holomorphic at point $z = \infty$, if in the neighbourhood of that point (for sufficiently large $|z|$) it may be expressed by a series of the form

$$a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + ...$$



Thus, from Eqn. (1) and Eqn. (2), $\Omega^*(z)$ and $\omega^*(z)$ may be represented outside L by the series as

$$\Omega^*(z) = \sum_{-\infty}^{\infty} a_n z^n, \tag{3}$$

$$\omega^*(z) = \sum_{-\infty}^{\infty} b_n z^n. \tag{4}$$

The component of stresses in complex form by Muskhelishvili (1963) are given by

$$\tau_{xx} + \tau_{yy} = 2[\Omega'(z) + \bar{\Omega}'(z)], \tag{5}$$

$$\tau_{yy} - \tau_{xx} + 2i\,\tau_{xy} = 2[\bar{z}\,\Omega''(z) + \omega'(z)]. \tag{6}$$

With the help of Eqn. (1) to Eqn. (4), Eqn. (5) to Eqn. (6) can be written as

$$\tau_{xx} + \tau_{yy} = 2\left\{-\frac{X+iY}{2\pi(1+\chi)}\frac{1}{z} - \frac{X-iY}{2\pi(1+\chi)}\frac{1}{\bar{z}} + \sum_{-\infty}^{\infty} n\left(a_n z^{n-1} + \bar{a}_n \bar{z}^{n-1}\right)\right\}, \tag{7}$$

$$\tau_{yy} - \tau_{xx} + 2i\,\tau_{xy} = 2\left[\bar{z}\frac{X+iY}{2\pi(1+\chi)}\frac{1}{z^2} + \frac{\chi(X-iY)}{2\pi(1+\chi)}\frac{1}{z} + \sum_{-\infty}^{\infty} n\left\{(n-1)\,a_n z^{n-2} + b_n z^{n-1}\right\}\right]. \tag{8}$$

For the boundedness of the stress, the last terms in the above expressions should be finite. Therefore we must have,

$$a_n = \bar{a}_n = 0, \quad (n \geq 2), \tag{9}$$

and

$$b_n = 0. \tag{10}$$

Thus, the boundary conditions at infinity are

$$\Omega(z) = -\frac{X+iY}{2\pi(1+\chi)}\log z + \Gamma z + \Omega_0(z), \tag{11}$$

$$\omega(z) = \frac{\chi(X-iY)}{2\pi(1+\chi)}\log z + \Gamma' z + \omega_0(z), \tag{12}$$

where $\Gamma = B + iC,$ (13)

$\Gamma' = B' + iC'$ (14)

are constants and $\Omega_0(z)$ and $\omega_0(z)$ are functions, holomorphic outside L including the point at infinity. For sufficiently large $|z|$, $\Omega_0(z)$ and $\omega_0(z)$ can be expressed as

$$\Omega_0(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + ..., \tag{15}$$

$$\omega_0(z) = b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + ... \tag{16}$$

The state of stress will not be altered by assuming $a_0 = b_0 = 0$. Thus, at infinity $\Omega_0(\infty) = \omega_0(\infty) = 0$. In Eqn. (13), C may be related to the rotation of an infinitely remote part of the plane and in the absence of rotation, $C = 0$.The real constants B, B' and C' in Eqn. (13) and Eqn. (14) have a very simple physical interpretation. These constants can be obtained from Eqn. (5) and Eqn. (6) for $z \to \infty$. Thus, at infinity

$$\tau_{xx} = 2B - B', \tag{17}$$

$$\tau_{yy} = 2B + B', \tag{18}$$

$$\tau_{xy} = C'. \tag{19}$$

Eqn. (17) to Eqn. (19) show that the stresses are uniformly distributed in the neighbourhood of the point at infinity. Let N_1 and N_2 be the value of the principal stresses at infinity and α the angle made by the direction of N_1 with x-axis, we have by Muskhelishvili (1963)

$$\tau_{xx} + \tau_{yy} = N_1 + N_2, \tag{20}$$

$$\tau_{yy} - \tau_{xx} + 2i\,\tau_{xy} = -(N_1 - N_2)\,e^{-2i\alpha}. \tag{21}$$

with the help of Eqn. (17) to Eqn. (21), we get

$$\Gamma = B = \frac{1}{4}(N_1 + N_2), \tag{22}$$

$$\Gamma' = B' + iC' = -\frac{1}{2}(N_1 - N_2)\,e^{-2i\alpha}. \tag{23}$$

III. SOLUTION OF THE PROBLEM

The component of stresses in polar coordinates are defined in the same way as in cartesian coordinates and is convenient to express. Converting Eqn. (5) and Eqn. (6) in polar coordinates:

$$\tau_{rr} + \tau_{\theta\theta} = 2\left[\phi(z) + \overline{\phi(z)}\right], \tag{24}$$

$$\tau_{\theta\theta} - \tau_{rr} + 2i\,\tau_{r\theta} = 2\,[\bar{z}\,\phi'(z) + \psi(z)]\,e^{2i\theta}. \tag{25}$$

From Eqn. (24) and Eqn. (25), we get

$$\tau_{rr} - i\,\tau_{r\theta} \;=\; \phi(z) \;+\; \overline{\phi(z)} \;-\; e^{2i\theta}\left[\bar{z}\,\phi'(z) + \psi(z)\right]. \tag{26}$$

Eqn. (26) can be written as

$$\tau_{rr} - i\,\tau_{r\theta} \;=\; N - iT, \tag{27}$$

where N and T are the components of the external stresses acting on the circumference of L in the direction of the normal, outward with respect to the body, and of the tangent directed to the left of the normal. Differentiating Eqn. (26), we get

$$\phi(z) \;=\; \Omega'(z) \;\text{ and }\; \psi(z) \;=\; \omega'(z),$$

$$\phi(z) \;=\; \sum_{k=0}^{\infty} a_k\, z^{-k}, \tag{28}$$

$$\psi(z) \;=\; \sum_{k=0}^{\infty} b_k\, z^{-k}, \tag{29}$$

where the coefficient a_0, a_1, b_0, b_1 have the values

$$a_0 \;=\; \Gamma \;=\; B, \tag{30}$$

$$b_0 = \Gamma' \;=\; B' + iC', \tag{31}$$

$$a_1 \;=\; -\frac{X + iY}{2\pi\,(1+\chi)}, \tag{32}$$

$$b_1 \;=\; \frac{\chi\,(X - iY)}{2\pi\,(1+\chi)}. \tag{33}$$

To use the condition of single valuedness of the displacement, using Eqn. (32) and Eqn. (33), we get

$$\chi a_1 \;+\; \bar{b}_1 \;=\; 0. \tag{34}$$

Using Eqn. (28) and Eqn. (29) to Eqn. (26) and assuming that the series converges to L, we get

$$\sum_{0}^{\infty} \frac{1+k}{R^k}\, a_k\, e^{-ik\theta} + \sum_{0}^{\infty} \frac{\bar{a}_k}{R^k}\, e^{ik\theta} - b_0 e^{2i\theta} - \frac{b_1}{R}\, e^{i\theta} - \sum \frac{a_{k+2}}{R^{k+2}}\, e^{-ik\theta}$$
$$=\; N - iT, \;\text{ on }\; L. \tag{35}$$

Expanding the function $N - iT$ given on L in a complex Fourier Series as

$$N - iT \;=\; \sum_{-\infty}^{\infty} A_k\, e^{ik\theta}. \tag{36}$$

From Eqn. (34) and Eqn. (35) comparing the coefficients of $e^{ik\theta}$

$$A_0 \;=\; 2a_0 \;-\; \frac{b_2}{R^2}, \tag{37}$$

$$A_1 = \frac{\bar{a}_1}{R} - \frac{b_1}{R}, \tag{38}$$

$$A_2 = \frac{\bar{a}_2}{R^2} - b_0, \tag{39}$$

$$A_n = \frac{\bar{a}_n}{R^n}, \quad (n \geq 3), \tag{40}$$

$$A_{-n} = \frac{1+n}{R^n} a_n - \frac{b_{n+2}}{R^{n+2}}, \ (n \geq 1). \tag{41}$$

As we know that a_0 and b_0 can be replaced by Γ and Γ', thus from Eqn. (39) we get

$$a_2 = \bar{\Gamma}' R^2 + \bar{A}_2 R^2. \tag{42}$$

The values of a_1 and b_1 can be obtained from Eqn. (38) with the help of Eqn. (34),

$$a_1 = \frac{\bar{A}_1 R}{1+\chi}, \qquad b_1 = -\frac{\chi A_1 R}{1+\chi}. \tag{43}$$

From Eqn. (41), we have

$$b_n = (n-1) R^2 a_{n-2} - R^n A_{-n+2}, \ (n \geq 3). \tag{44}$$

Let us suppose the edges of the hole be free from external stresses and P is the tension in direction of x-axis such that

$$\tau_{xx} = P = N_1, \ \tau_{yy} = N_2 = 0, \ \tau_{xy} = 0. \tag{45}$$

With the help of Eqn. (22) and Eqn. (23), we get

$$\Gamma = \frac{P}{4}, \qquad \Gamma' = -\frac{P}{2}. \tag{46}$$

On the circle $N - iT = 0$, thus $A_k = 0$ in the Eqn. (36) and from Eqn. (40) to Eqn. (44), we get

$$a_n = 0, \qquad (n \geq 3), \tag{47}$$

$$a_n' = 0, \qquad (n \geq 5). \tag{48}$$

From Eqn. (22), Eqn. (23), Eqn. (43) and Eqn. (44) we get

$$a_0 = \frac{P}{4}, \qquad a_1 = 0, \qquad a_2 = -\frac{PR^2}{2}, \tag{49}$$

$$b_0 = -\frac{P}{2}, \qquad b_1 = 0, \qquad b_2 = \frac{PR^2}{2}, \tag{50}$$

$$b_3 = 0, \qquad b_4 = -\frac{3PR^4}{2}. \tag{51}$$

Thus from Eqn (28) and Eqn. (29) we get expressions,

$$\phi(z) = \frac{P}{4}\left(1 - \frac{2R^2}{z^2}\right), \tag{52}$$

$$\psi(z) = -\frac{P}{2}\left(1 - \frac{R^2}{z^2} + \frac{3R^4}{z^4}\right). \tag{53}$$

With the help of Eqn (52) and Eqn. (53) expression for stress components can be found out and the problem is solved completely.

IV. EXPRESSIONS FOR STRESSES

Expression for stresses in the polar coordinates can be determined from Eqn. (24) and Eqn. (25) by putting $z = re^{i\theta}$. Thus we find expressions

$$\tau_{rr} + \tau_{\theta\theta} = P\left(1 - \frac{2\,R^2}{r^2}\cos 2\theta\right), \tag{54}$$

$$\tau_{\theta\theta} - \tau_{rr} + 2i\,\tau_{r\theta} = P\left\{\frac{2R^2}{r^2}e^{-2i\theta} - e^{2i\theta} + \frac{R^2}{r^2} - \frac{3R^4}{r^4}e^{-2i\theta}\right\}. \tag{55}$$

Separating real and imaginary parts and solving for τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{r\theta}$ we get

$$\tau_{rr} = \frac{P}{2}\left(1 - \frac{R^2}{r^2}\right) + \frac{P}{2}\left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4}\right)\cos 2\theta, \tag{56}$$

$$\tau_{\theta\theta} = \frac{P}{2}\left(1 + \frac{R^2}{r^2}\right) - \frac{P}{2}\left(1 + \frac{3R^4}{r^4}\right)\cos 2\theta, \tag{57}$$

$$\tau_{r\theta} = -\frac{P}{2}\left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4}\right)\sin 2\theta. \tag{58}$$

V. RESULTS AND DISCUSSION

Variation of the stress components $\tau_{rr}, \tau_{\theta\theta}$ and $\tau_{r\theta}$ have been studied graphically in Fig. (2) to Fig. (4). From the expression of stresses it is clear that for $R/r = 1$, i.e., on L, $\tau_{rr} = \tau_{r\theta} = 0$ and the value of tensile stress $\tau_{\theta\theta}$ is given by $\tau_{\theta\theta} = P(1 - 2\cos 2\theta)$.

The distribution of the stress component τ_{rr} has been shown in Fig. (2), while in Fig. (3), the distribution of the stress component $\tau_{\theta\theta}$ has been shown.The stresses are uniformly distributed in the neighbourhood of the point at infinity. From Fig. (3), the maximum value of $\tau_{\theta\theta}$ is at $\theta = -\pi/2$ and $\pi/2$. The maximum value of $\tau_{\theta\theta}$ is $3P$, which shows that the value of the tensile stress is increased.

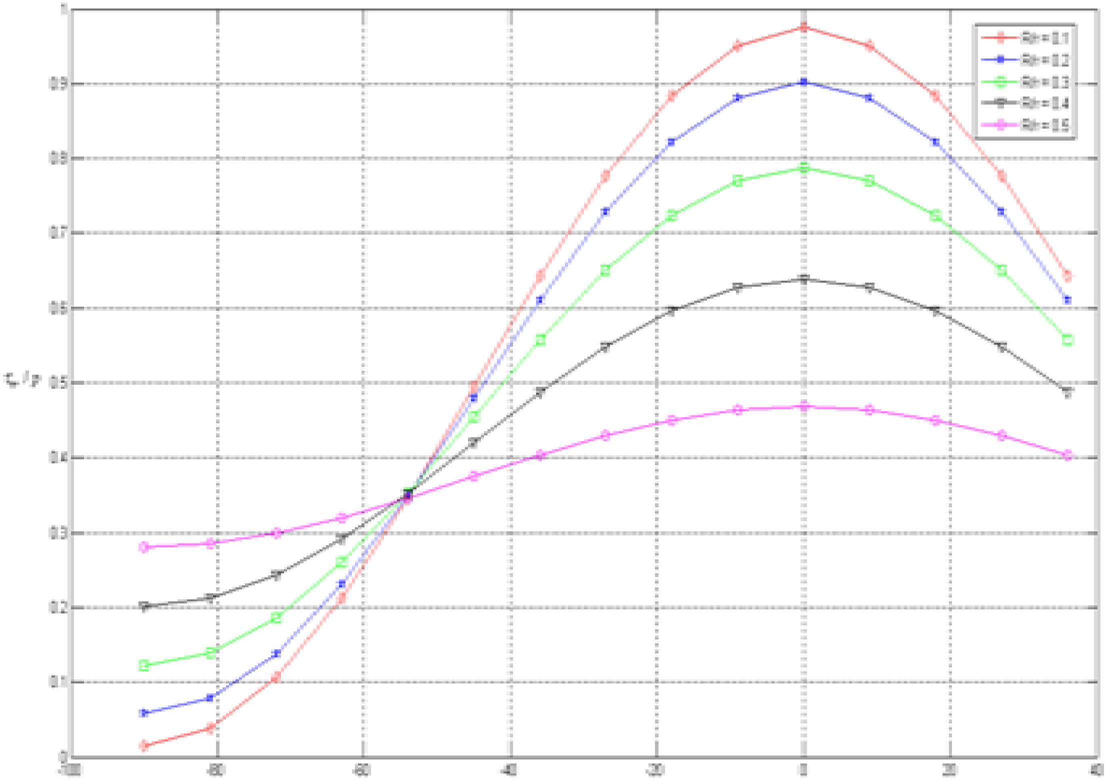


Fig. 2 Distribution of the Stress Component τ_{rr} Round the Edge of a Circular Hole

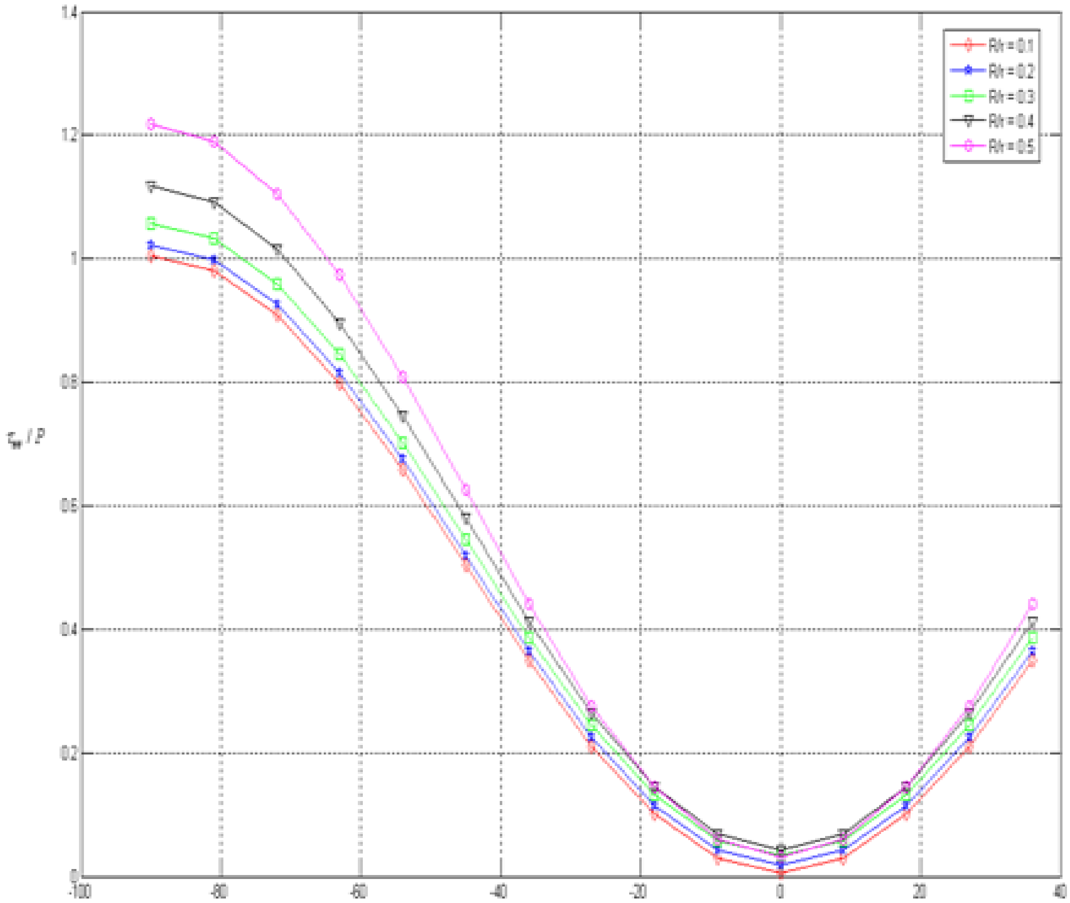


Fig. 3 Distribution of the Stress Component $\tau_{\theta\theta}$ Round the Edge of a Circular Hole

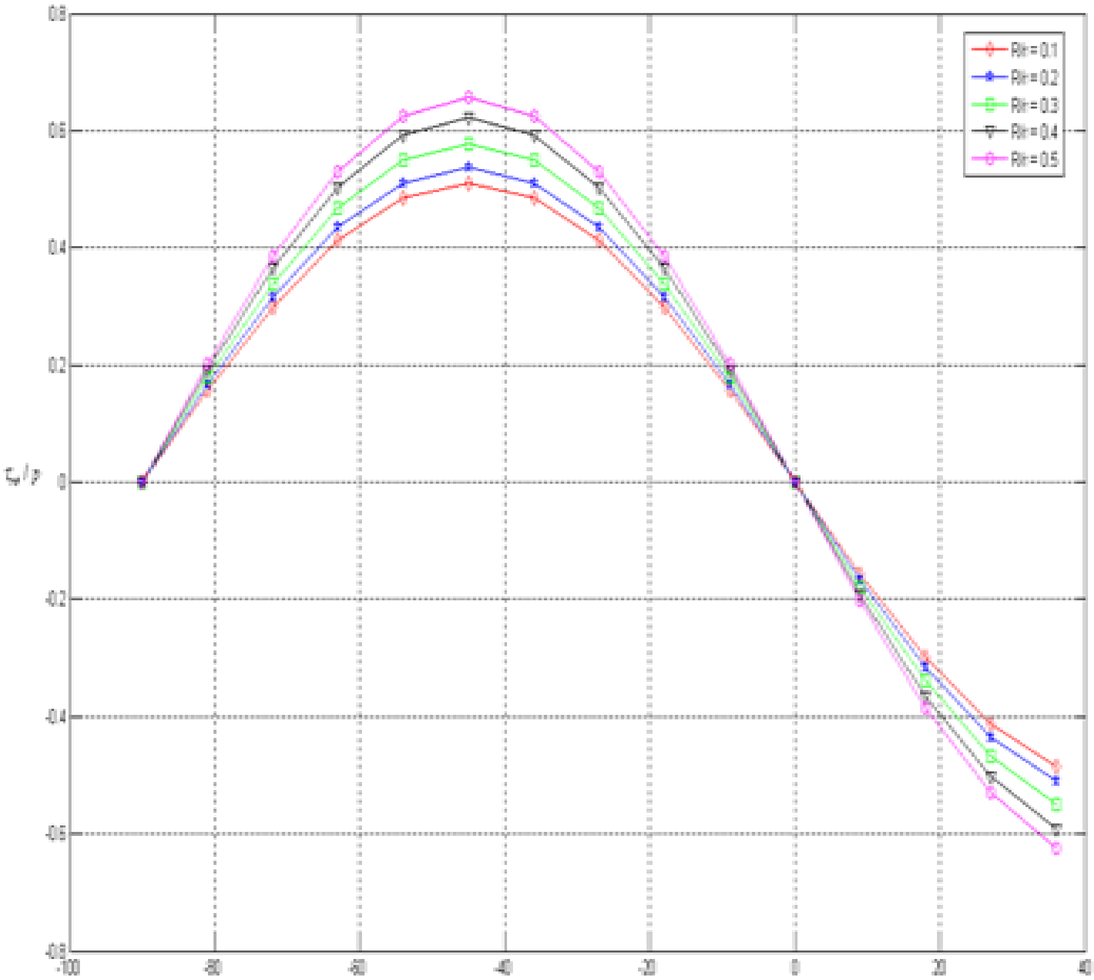


Fig. 4 Distribution of the Stress Component $\tau_{r\theta}$ Round the Edge of a Circular Hole

VI. CONCLUSION

The investigations presented in this work are concerned with unidirectional tension of an infinite plate weakened by a circular hole. The edges of the hole are free from external stresses. Results obtained in this problem can be employed for other similar problems. The solution of the problem of bi-axial tension of an infinite plate with a circular hole can be obtained directly from this work by superimposing two uni-directional stress distribution along the axes OX and OY respectively. One can also investigate the same problem by applying by an uniform normal pressure or a concentrated force or a concentrated couple to the edges of the hole.

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